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Pseudo-Euclidean Hurwitz pairs, sigma and pure spinor models

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Abstract. We consider the pseudo-Euclidean Hurwitz pairs, as formulated in a recent study by Randriamihamison. From these pairs, we construct solutions of sigma models defined on real hyperquadrics and solutions of pure spinor models. This provides a non-trivial extension of previous works of Fujii.

1. Introduction

Starting from the work of Lawrinowicz and Rembielinski (1987), Randriamihamison (1990) has given an explicit method to generate all pseudo-Euclidean Hurwitz pairs. This approach uses the spinorial formalism developed by Crumeyrolle (1989). We recall that $\{(F, Q_0), (S, \Lambda)\}$ is a pseudo-Euclidean pair of dimension (m, s) (m and s are integers and s is even) if and only if:

(a) F is a real vector space of dimension m endowed with a quadratic degenerate pseudo-Euclidean form Q_0 of signature (ρ, σ), $\rho \geqslant 1$, $\sigma \geqslant 0$,

(b) S is a real vector space of dimension s endowed with a non-degenerate, symmetric or antisymmetric bilinear form Λ;

(c) there exists a bilinear map $f: F \times S \to S$ such that:

(i) there exists a unique element ε_0 of F such that:

$$\forall \varphi \in S \qquad f(\varepsilon_0, \varphi) = \varphi$$

(ii) $\forall a \in F$, $\forall \varphi, \psi \in S$:

$$\Lambda(a\varphi, a\psi) = Q_0(a)\Lambda(\varphi, \psi)$$

where $f(a\varphi) = a\varphi$;

(iii) the action of F on S is irreducible, i.e. it does not exist a proper subspace of S remaining stable under the action of F on S defined by f.

A pseudo-Euclidean Hurwitz pair $\{(F, Q_0), (S, \Lambda)\}$ is connected with a real Clifford algebra $C(Q)$, where Q is the non-degenerate quadratic form such that $Q(x) = -Q_0(x)$ for any x in the subspace E defined by $F = \mathbb{R}\varepsilon_0 \oplus E$. Following Randriamihamison (1990) S can be viewed as the spinor space associated with $C(Q)$. The real matrices $e_1, e_2, \ldots, e_{m-1}$ generating a basis of $C(Q)$ must satisfy the following constraint (Lawrinowicz and Rembielinski 1987, Randriamihamison 1990):

$$e_j L = -L e_j^T \qquad (j = 1, 2, \ldots, m-1)$$

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where L is the matrix representation of Λ in the basis of the spinor space S. The action of F on S can now be written as follows:

$$f(a, \varphi) = a\varphi \equiv \left(a^0 \mathbb{1}_s + \sum_{j=1}^{m-1} a^j e_j\right)\varphi$$

where $a^0, a^1, \ldots, a^{m-1}$ are real numbers. In the work of Randriamihamison (1990) we find all the dimensions (m, s) for which a pseudo-Euclidean Hurwitz pair exists as well as the signatures (ρ, σ) of Q_0 and the nature of the corresponding Λ (symmetric or antisymmetric).

Fujii (1985) has given examples of generators P and U of reduced K-group on S^{2n} or S^{2n-1} (see Atiyah, 1967) which are also harmonic maps i.e. solutions of sigma models. More precisely he starts with $2n-1$ generators $E_1, E_2, \ldots, E_{2n-1}$ of a Euclidean complex Clifford algebra. Each generator belongs to the unitary group $\mathrm{O}(N)$, where $N = 2^{n-1}$. Let $G(2N, N; \mathbb{C})$ be the Grassmannian manifold defined by

$$G(2N, N; \mathbb{C}) = \{P \in C(2N) \text{ such that } P^2 = P, P^+ = P, \operatorname{tr} P = N\}$$

where $C(2N)$ denotes the algebra of $2N \times 2N$ complex matrices. This manifold can be identified with the symmetric space

$$\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N).$$

Let us endow the sphere $S^{2n-1}(S^{2n})$ with the stereographic coordinates $x^1, x^2, \ldots, x^{2n-1}(x^{2n})$. Then, the map P defined by

$$P: \qquad S^{2n} \to G(2N, N; \mathbb{C}) : (x^1, \ldots, x^{2n}) \to P(x^1, \ldots, x^{2n})$$

where

$$P(x^1, \ldots, x^{2n}) = \frac{1}{1+|z|^2} \begin{bmatrix} \mathbb{1}_N & Z^+ \\ Z & |Z|^2 \mathbb{1}_N \end{bmatrix} \qquad Z = x^{2n} \mathbb{1}_N + \mathrm{i} \sum_{j=1}^{2n-1} x^j E_j \tag{1}$$

and

$$|Z|^2 = \sum_{j=1}^{2n} (x^j)^2$$

satisfies the equation

$$\left[P, \partial_j \left\{\frac{1}{(1+|Z|^2)^{2n-2}} \partial^j P\right\}\right] = 0 \qquad \partial_j = \frac{\partial}{\partial x^j}$$

which is nothing but the field equation of a sigma model defined on S^{2n} and with values on $G(2N, 2; \mathbb{C})$. Similarly it is possible to introduce the map

$$\mathrm{U}: \qquad S^{2n-1} \to \mathrm{U}(N) : (X^1, \ldots, X^{2n-1})$$

where

$$\mathrm{U}(X^1, \ldots, X^{2n-1}) = \frac{1}{1+|w|^2} \left\{(1-|w|^2)\mathbb{1}_N + 2\mathrm{i} \sum_{j=1}^{2n-1} x^j E_j\right\} \qquad |w|^2 = \sum_{j=1}^{2n-1} (x^j)^2. \tag{2}$$

This map satisfies the equation,

$$\partial_j \left\{\frac{1}{(1+|w|^2)^{2n-3}} \mathrm{U}^+ \partial^j \mathrm{U}\right\} = 0$$

which is the field equation of a principal sigma model defined on S^{2n-1} and with values on $\mathrm{U}(N)$.

In another work, Fujii (1988) studies the solutions of the field equation of a pure spinor model:

$$i\not{\partial}\psi+\frac{n}{n-1}g(\psi^{+}\psi)^{1/n-1}\psi=0$$

where $n\neq 1$, g is a constant and

$$\not{\partial}=\sum_{j=1}^{n}E_j\partial_j$$

with $E_1,\ldots,E_n$ a set of matrices of $\mathrm{U}(N)$, $N=2^p$ and $p=E((n-2)/2)$, generating a Euclidean complex Clifford algebra. Fujii has given the following solutions belonging to the spinor space associated with this algebra:

$$\psi=\frac{1}{(a^2+|v|^2)^{n/2}}\left(a\mathbb{1}_N+i\sum_{j=1}^{n}x^jE_j\right)\psi_0 \qquad |v|^2=\sum_{j=1}^{n}(x^j)^2 \tag{3}$$

where a is an arbitrary constant and ψ_0 is a constant spinor given by,

$$(\psi_0^+\psi_0)^{1/n-1}=(n-1)\frac{a}{g}.$$

Another solution can be written as follows:

$$\psi=\frac{1}{(|v|^2)^{(n-1)/2}}\left\{1\pm\frac{1}{(|v|^2)^{1/2}}i\sum_{j=1}^{n}x^jE_j\right\}\psi_0 \tag{4}$$

where ψ_0 is a constant spinor satisfying

$$(2\psi_0^+\psi_0)^{1/n-1}=\frac{(n-1)^2}{2n}\frac{1}{g}.$$

The aim of this paper is to find a real generalization of the solutions of Fujii: (1), (2), (3) and (4) with a unified algebraic framework: the theory of pseudo-Euclidean Hurwitz pairs. Namely, this framework provides a tool to generate new harmonic maps defined on real hyperquadrics and new solutions of pure real spinor models. The above quoted generalization is a non-trivial one because not all of these maps can be derived from the solutions of Fujii.

2. Pseudo-Euclidean Hurwitz pairs and harmonic maps

Let (F, Q_0), (S, Λ) be an arbitrary pseudo-Euclidean Hurwitz pair of dimension (m, s). Let K be the matrix representing Q_0 in a pseudo-orthonormal basis of F. Using the same notations as above and following the paper of Lawrinowicz and Rembielinski (1987) it is always possible to find a matrix L such that

$$L^T=\delta L \qquad L^2=\delta\mathbb{1}_s \qquad \delta=\pm 1$$

and then

$$e_j^T=-\delta L e_j L$$

for each e_j $(j=1,\ldots,m-1)$ generating a real matrix representation of $C(Q)$.

Let A be the element of real Clifford algebra $C(Q)$ defined by

$$A=a^0\mathbb{1}_s+\sum_{j=1}^{m-1}a^je_j.$$

Let a be the real vector of F defined by $a^T=(a^0, a^1, \ldots, a^{m-1})$. We get the following results

$$L\bar{A}=A^TL \qquad \bar{A}=a^0\mathbb{1}_s-\sum_{j=1}^{m-1}a^je_j \tag{5}$$

and

$$A^TLA=Q_0(a)L. \tag{6}$$

Equation (6) is what we called, in a previous work (Lambert and Kibler, 1988), the Hurwitz property (when $m=s=2,4,8$). It is now possible to introduce the $2s\times s$ matrix Z and the $2s+2s$ matrix M given by

$$Z=\begin{bmatrix}\mathbb{1}_s\\ A\end{bmatrix} \qquad M=\begin{bmatrix}L & 0\\ 0 & L\end{bmatrix}. \tag{7}$$

The $2s\times 2s$ real matrix $P(a)$ defined, for each a in F, by

$$P(a)=Z(Z^TMZ)^{-1}Z^TM \tag{8}$$

satisfies the equations

$$P(a)^2=P(a) \qquad P(a)^T=\delta MP(a)M. \tag{9}$$

Using (5), (6) and (7) we rewrite (8) as follows:

$$P(a)=\frac{1}{1+Q_0(a)}\begin{bmatrix}\mathbb{1}_s & \bar{A}\\ A & Q_0(a)\mathbb{1}_s\end{bmatrix}. \tag{10}$$

Equations (9) shows that $P(a)$ parametrizes a point on the symmetric space $\mathrm{O}(M)/\mathrm{O}(L)\times\mathrm{O}(L)$, where $\mathrm{O}(L)$ denotes the classical group leaving L invariant. When $\delta=+1$, this group is a pseudo-orthogonal one. When $\delta=-1$, it is a symplectic one.

Let $x^0, x^1, \ldots, x^{m-1}, x^m$ be real variables. The equations $(x^m)^2\pm Q_0(x)=1$, where x is the vector defined by $x^T=(x^0, \ldots, x^{m-1})$, defines the hyperquadrics $H_\pm(\rho,\sigma)$ of dimension $m=\rho+\sigma$. These hyperquadrics can be identified with the following symmetric spaces:

$$H_+(\rho,\sigma)=\mathrm{SO}(\rho+1,\sigma)/\mathrm{SO}(\rho,\sigma) \qquad (\rho\geqslant 1, \sigma\geqslant 0)$$

and

$$H_-(\rho,\sigma)=\mathrm{SO}(\sigma+1,\rho)/\mathrm{SO}(\sigma,\rho) \qquad (\rho\geqslant 1, \sigma\geqslant 0).$$

Furthermore, we get the following homeomorphisms:

$$\begin{aligned}
&H_+(\rho,0)\simeq S^\rho && (\rho\geqslant 1)\\
&H_+(\rho,\sigma)\simeq\mathbb{R}^\sigma\times S^\rho && (\rho \text{ and } \sigma\geqslant 1)\\
&H_-(\rho,\sigma)\simeq\mathbb{R}^\rho\times S^\sigma && (\rho \text{ and } \sigma\geqslant 1)\\
&H_-(\rho,0)\simeq\mathbb{R}^\rho\times\mathbb{Z}_2 && (\rho\geqslant 1).
\end{aligned}$$

The last is in fact the hyperbolic space of dimension ρ. Let $a^0, \ldots, a^{m-1}$ be stereographic coordinates on $H_\pm(\rho,\sigma)$. Then equation (10) leads to the maps:

$$P_\pm: \qquad H_\pm(\rho,\sigma)\to\mathrm{O}(M)/\mathrm{O}(L)\times\mathrm{O}(L): a\to P_\pm(a)$$

where

$$P_\pm(a)=\frac{1}{1\pm Q_0(a)}\begin{bmatrix}\mathbb{1}_s & \bar{A}\\ a & \pm Q_0(a)\mathbb{1}_s\end{bmatrix} \tag{11}$$

with $Q_0(a) \neq \mp 1$. We now prove that $P_\pm$ define harmonic maps or equivalently solutions of sigma models. With the above stereographic coordinates we are able to define the Laplace-Beltrami operators on $H_\pm(\rho, \sigma)$, given by

$$\Delta_\pm = (1 \pm Q_0(a))^m \partial_j \left(\frac{1}{(1 \pm Q_0(a))^{m-2}} K^{jk} \partial_k \right) \tag{12}$$

for any a such that $Q_0(a) \neq \mp 1$ and $\partial_j = \partial/\partial a^j$ $(j = 0, \ldots, m-1)$. The maps $P_\pm$ are the harmonic if and only if (see Eells and Lemaire 1988)

$$[P_\pm, \Delta_\pm P_\pm] = 0. \tag{13}$$

Using (5), (11) and (12) we get

$$\Delta_\pm P_\pm = \frac{(-2m)}{1 \pm Q_0(a)} \begin{bmatrix} (1 \mp Q_0(a))\mathbb{1}_s & 2\bar{A} \\ 2\bar{A} & -(1 \mp Q_0(a)) \end{bmatrix}. \tag{14}$$

Equations (11) and (14) lead to the result (13).

Let $h_\pm(\rho, \tau)$ the hyperquadrics define by $Q_0(x) = \pm 1$. These hyperquadrics can be identified with the following symmetric spaces:

$$h_+(\rho, \sigma) = \mathrm{SO}(\rho, \sigma)/\mathrm{SO}(\rho - 1, \sigma) \qquad (\rho \geqslant 1, \sigma \geqslant 0)$$

and

$$h_-(\rho, \sigma) = \mathrm{SO}(\sigma, \rho)/\mathrm{SO}(\sigma - 1, \rho) \qquad (\rho \geqslant 1, \sigma \geqslant 0).$$

We also have the homeomorphisms:

$$\begin{aligned} & h_+(\rho, 0) \simeq S^{\rho-1} && (\rho \geqslant 1) \\ & h_+(\rho, \sigma) \simeq \mathbb{R}^\sigma \times S^{\rho-1} && (\rho \text{ and } \sigma \geqslant 1) \\ & h_-(\rho, \sigma) \simeq \mathbb{R}\rho \times S^{\sigma-1} && (\rho \text{ and } \sigma > 1) \\ & h_-(\rho, 1) \simeq \mathbb{R}^\rho \times \mathbb{Z}_2 && (\rho \geqslant 1). \end{aligned}$$

Let $b^1, \ldots, b^{m-1}$ be a set of stereographic coordinates on $h_\pm(\rho, \sigma)$ such that $Q(b) \neq \pm 1$, $b \in E$ and $b^T = (b^1, \ldots, b^{m-1})$. Then we are able to introduce the maps $\mathrm{U}_\pm$ given by

$$\mathrm{U}_\pm: \qquad h_\pm(\rho, \sigma) \to \mathrm{O}(L): b \to \mathrm{U}_\pm(b)$$

with

$$\mathrm{U}_\pm(b) = \frac{1}{1 \mp Q(b)} \left\{ (1 + Q(b))\mathbb{1}_s + 2 \sum_{j=1}^{m-1} b^j e_j \right\}. \tag{15}$$

The fact that $\mathrm{U}_\pm(b)$ belong to $\mathrm{O}(L)$ can be obviously checked from the properties of the matrix e_j. The maps $\mathrm{U}_\pm$ are harmonic maps if and only if (see Eells and Lemaire 1988):

$$\partial_j \left(\frac{1}{(1 \mp Q(b))^{m-3}} \mathrm{U}_\pm(b)^T L \partial^j \mathrm{U}_\pm(b) \right) = 0 \qquad (j = 1, \ldots, m-1). \tag{16}$$

Using (15), a straightforward computation leads to (16). Thus $\mathrm{U}_\pm$ are harmonic maps.

3. Pseudo-Euclidean Hurwitz pairs and pure spinor models

If (F, Q_0), (S, Λ) is a pseudo-Euclidean Hurwitz pair of dimension (m, s), $m>2$, then S is a space of real spinors associated with the real Clifford algebra $C(Q)$. Starting with a real vector a of F, $Q_0(a)>0$, $a^T=(a^0, a^1, \ldots, a^{m-1})$ and assuming that a^0 remains constant, it is possible to define a spinor field on a domain $D_0 \subseteq E$ (using the notations of sections 1 and 2):

$$\psi: \qquad D_0 \subseteq E \to S: a \to \psi(a)$$

where

$$\psi(a)=\frac{A}{(Q_0(a))^{(m-1)/2}}\,\psi_0 \tag{17}$$

with ψ_0 a constant spinor in S such that

$$(\psi_0^T L \psi_0)^{1/m-2}=(m-2)\,\frac{a^0}{g} \qquad (g \neq 0, \text{ real constant}).$$

It is straightforward to check the following result:

$$(\psi(a)^T L \psi(a))^{1/m-2}=\frac{(m-2)}{Q_0(a)}\,\frac{a^0}{g}. \tag{18}$$

We define the Dirac operator on E by the equation

$$\not{\partial}=\sum_{j=1}^{m-1} e_j \partial^j \qquad \partial^j=K^{jl}\partial_l.$$

Using (17) we get

$$\not{\partial}\psi(a)=-(m-1)a^0\frac{A}{(Q_0(a))^{(m+1)/2}}\,\psi_0. \tag{19}$$

Equations (18) and (19) show that the spinor field $\psi(a)$ satisfies

$$\not{\partial}\psi(a)+\left(\frac{m-1}{m-2}\right)g(\psi(a)^T L\psi(a))^{1/m-2}\psi(a)=0 \tag{20}$$

which is the field equation of a pure real spinor model defined on $D_0=\{a \in F$ such that a^0 is constant an $Q_0(a)>0\}$ and with values on S.

Another set of solutions of equation (20) can be obtained defining spinor fields on a domain $D \subseteq E$:

$$\psi_\pm: \qquad D \subseteq E \to S: b \to \psi(b)$$

where b is a real vector in E such that $b^T=(b^1, \ldots, b^{m-1})$ and $Q(b)<0$,

$$\psi_\pm(b)=\frac{1}{(-Q(b))^{m-2/4}}\left(\mathbb{1}_s \pm \frac{1}{(-Q(b))^{1/2}}\sum_{j=1}^{m-1} b^j e_j\right)\psi_0 \tag{21}$$

where ψ_0 is a constant spinor defined with a real constant $g \neq 0$:

$$(2\psi_0^T L\psi_0)^{1/(m-2)}=\frac{(m-2)^2}{2g(m-1)}. \tag{22}$$

We immediately check that the equation

$$(\psi(b)^T L\psi(b))^{1/(m-2)}=\frac{(m-2)^2}{2g(m-1)}\,\frac{1}{(-Q(b))^{1/2}}. \tag{23}$$

Let us compute $\partial\psi_\pm(b)$, we obtain

$$\partial\psi_\pm(b)=\frac{-(m-2)}{2(-Q(2))^{m/2}}\left(\frac{1}{(-Q(b))^{1/2}}\sum_{j=1}^{m-1} b^j e_j \pm 1\right)\psi_0. \tag{24}$$

Equations (21), (23) and (24) show that (20) is satisfied. Thus $\psi_\pm(b)$ are solutions of a pure real spinor model defined on $D=\{b\in E$ such that $Q(b)<0\}$ and with values on S.

Equation (20) defines a pure real spinor model which exhibits general invariance property. The scalar product $\psi(a)^T L\psi(a)$ defined from the bilinear form Λ happens to be invariant under the transformation

$$\psi(a)\to c\psi(a)$$

where c is an arbitrary element of the reduced Clifford group G_0^+ (see Crumeyrolle 1990). It is now obvious to prove that equation (20) remains invariant under the action of G_0^+. The theory of Clifford algebras leads to the following results (Crumeyrolle 1990):

(a) When $(m-1)$ is even then:
if Q is positive definite: $\mathrm{Spin}(Q)=G_0^+$
if Q is negative definite: $\mathrm{Spin}(Q)=G_0^+$
if Q is indefinite: $G_0^+\subset \mathrm{Spin}(G)$
(b) When $(m-1)$ is odd then:
if Q is definite: $\mathrm{Spin}(Q)=G_0^+$
if Q is indefinite: $G_0^+\subset\mathrm{Spin}(Q)$.
Thus equation (20) leads to $\mathrm{Spin}(Q)$-invariant real models when Q is definite.

4. Pseudo-Euclidean Hurwitz pairs and solutions of Fujii

In this section we show that the solutions (11), (15), (17), (21) cannot be obtained from the solutions (1), (2), (3), (4) when $2n-1\neq 3$ or $7 \pmod 8$.

The key point of Fujii's construction is the existence of a (2^{n-1})-dimensional complex representation of the Euclidean complex Clifford algebra with $2n-1$ generators, in $\mathrm{U}(2^{n-1})$. In order to construct a Hurwitz pair $\{(F, Q_0), (S, \Lambda)\}$ related with *real* Euclidean Clifford algebra $C(Q)$ generated by matrices $e_1, \ldots, e_{2n-1}$ $(e_j^2=\mathbb{1})$, Randriamihamison (1990) considers the following cases:

case (a) $2n-1=1 \pmod 8$ and $n=1=1 \pmod 2$
case (b) $2n-1=5 \pmod 8$ and $n-1=1 \pmod 2$
case (c) $2n-1=3, 7 \pmod 8$.

We define $C(Q)'$ as the complexified Clifford algebra of $C(Q)$: $C(Q)'=C(Q)\otimes\mathbb{C}$. Let f be a primitive idempotent of $C(Q)'$ and let re S' the realified space of $S'=C(Q)'f$. Then we have:

In case (a): (i) $S=S_1$, where S_1 is the space of Majorana spinors of $S'=C(Q)f$
(ii) $\dim_{\mathbb{R}} S_1=2^{n-1}$ and $C(Q)\simeq\mathbb{R}(2^{n-1})\oplus\mathbb{R}(2^{n-1})$

In case (b): (i) $S=\mathrm{re}\, S'$
(ii) $\dim_{\mathbb{R}} S=2^n$ and $C(Q)\simeq\mathbb{H}(2^{n-2})\oplus\mathbb{H}(2^{n-2})$

In case (c): (i) $S=\mathrm{re}\, S'$
(ii) $\dim_{\mathbb{R}} S=2^n$ and $C(Q)\simeq\mathbb{C}(2^{n-2})$.

It is now obvious to check that the solutions generated by Hurwitz pairs can be obtained from Fujii's one passing from complex numbers to real numbers if and only if $2n-1=3$ or $7 \pmod 8$.

5. Application

Let us start with a pseudo-Euclidean Hurwitz pair $\{(F, Q_0), (S, \Lambda)\}$, where $F=\mathbb{R}^{2,2}$ and Q_0 with signature $(\rho, \sigma)=(2, 2)$. Following Randriamihamison (1990) we find $S=S_1$, the space of Majorana spinors of $C(Q)'$, and

$$L=\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

We choose e_1, e_2 and e_3 as the real matrices:

$$e_1=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad e_2=\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad e_3=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We check the property quoted in section 1:

$$Le_j^T=-e_jL \qquad (j=1, 2 \text{ and } 3).$$

Then, with $s=4$, we get the following harmonic maps:

$$P_{\pm}: \qquad SO(3,2)/SO(2,2) \rightarrow Sp(8,\mathbb{R})/Sp(4,\mathbb{R}) \times Sp(4,\mathbb{R})$$

$$U_{\pm}: \qquad SO(2,2)/SO(1,2) \rightarrow Sp(4,\mathbb{R})$$

and we obtain solutions of real pure spinor models:

$$\psi: \qquad D_0 \rightarrow S_1 \simeq \mathbb{R}^4$$

where $D_0=\{a \in \mathbb{R}^{2,2},\ a^T=(a^0, a^1, a^2, a^3)$ such that a^0 is a real constant and $(a^0)^2-(a^1)^2-(a^2)^2+(a^3)^2>0\}$

$$\psi_{\pm}: \qquad D \rightarrow S_1 \simeq \mathbb{R}^4$$

where $D=\{b \in \mathbb{R}^{1,3},\ b^T=(b^1, b^2, b^3)$ such that $(b^1)^2+(b^2)^2-(b^3)^2<0\}$. Thus $\psi_{\pm}$ defines Majorana spinors in the upper cone of a three-dimensional Minkowskian space.

In a forthcoming paper, we discuss the applications of the spinor solutions described above in the theory of axisymmetric gravitational fields in general relativity. For the axially symmetric fields, Ernst's equations are integrable. But the solutions of these equations are connected with complex symmetric spaces which arises naturally in the theory of pseudo-Euclidean Hurwitz pairs (see Mazur 1983, Hogan 1984).

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